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# Two-dimensional metric and tetrad gravities as constrained second order systems

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## Abstract

Using the Gitman-Lyakhovich-Tyutin generalization of the Ostrogradsky method for analyzing singular systems, we consider the Hamiltonian formulation of metric and tetrad gravities in two-dimensional Riemannian spacetime treating them as constrained higher-derivative theories. The algebraic structure of the Poisson brackets of the constraints and the corresponding gauge transformations are investigated in both cases.

*Keywords:* Hamiltonian formulation, Einstein-Hilbert action, metric and tetrad gravities, two dimensions, Ostrogradsky method.

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A well known major difference between gravity and other field theories is that the former includes second order time derivatives in its usual Lagrangian formulation. In order to pass to a Hamiltonian formulation, Pirani *et al.* [1], [2] circumvented this situation by subtracting a divergence term from the Lagrangian which includes second order time derivatives, leaving one with first order time derivatives of the fields in the Lagrangian and making it suitable for the Dirac treatment of constrained systems [3].<sup>1</sup> More specifically, they split the  $d$ -dimensional Einstein-Hilbert (EH) action in the following way<sup>2</sup>

$$S_{EH} = \int d^d x \sqrt{(-1)^{d-1} g} R = \int d^d x L_{\Gamma\Gamma} - \int d^d x V_{,\alpha}^\alpha, \quad (1)$$

where

$$\begin{aligned} L_{\Gamma\Gamma} &= \sqrt{(-1)^{d-1} g} g^{\mu\nu} \left( \Gamma_{\sigma\mu}^\lambda \Gamma_{\lambda\nu}^\sigma - \Gamma_{\sigma\lambda}^\lambda \Gamma_{\mu\nu}^\sigma \right), \\ V^\alpha &= \sqrt{(-1)^{d-1} g} \left( g^{\alpha\mu} \Gamma_{\mu\nu}^\nu - g^{\mu\nu} \Gamma_{\mu\nu}^\alpha \right), \end{aligned}$$

and then neglect the contribution of the surface term  $V_{,\alpha}^\alpha$  as its inclusion has no effect on equations of motion. As was emphasized in [1], a problem with this approach is that the part of the action remaining after elimination of the surface term, the  $\Gamma\Gamma$ -part, is not invariant under a general coordinate transformation. This can lead to inconsistencies in the Hamiltonian treatment of this system.

The importance of a surface term for the Hamiltonian formulation of General Relativity based on the ADM slicing of spacetime was emphasized by Regge and Teitelboim [6] and in the path-integral approach by Hawking [7].

In two dimensions the gravitational action, given in (1), is a topological quantity. The equations of motion for  $g_{\mu\nu}$  in two dimensions are trivial identities, putting no restriction on the metric. No matter how the Cauchy problem is formulated, the gravitational fields are arbitrary functions of spacetime; as any possible configuration of metric extremizes the action, the latter is a constant.

However, in two dimensions not only does the  $L_{\Gamma\Gamma}$  part of (1) not vanish [8], but it also gives rise to a striking but consistent Hamiltonian treatment of two dimensional gravity.

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<sup>1</sup> Another alternative way to circumvent this situation is by working with first order formalisms as briefly discussed later.

<sup>2</sup>  $\Gamma_{\sigma\mu}^\lambda$  is the Christoffel symbol,  $g = \det g_{\mu\nu}$ , the signature is  $(+, -, -, \dots)$ , and the convention for defining the Riemann tensor is the one used in [4] or [5].

Applying the Dirac formalism to the  $L_{\text{TF}}$  leads to a gauge transformation [8] which is simply that one can add any arbitrary tensor to the metric tensor. This is consistent with the metric tensor being arbitrary.

In contrast, the total two-dimensional Lagrangian of (1), when expressed in terms of tetrad variables, is a total derivative. This fact has erroneously led to the conclusion that a Hamiltonian formulation of two-dimensional gravity is impossible.<sup>3</sup> However, a Hamiltonian treatment of (1) is possible in the metric formulation on account of the inequivalence of the metric and tetrad formulations of gravity [9].<sup>4</sup>

One approach to the Hamiltonian formulation of gravity which avoids the problem of second order time derivatives is to work with the first order formulations of gravity, such as the metric-affine connection formulation of Einstein [10], or the tetrad-spin connection formulation [11] - [14].

For the two-dimensional case, however, the equivalence of the metric-affine connection or tetrad-spin connection formulation with the original second order formulations is obscured, as the metric (tetrad) no longer uniquely determines the affine (spin) connection. This is explained in [15] - [17]. Fortunately, there is still one more approach that is applicable to theories with higher-order time derivatives which involves reducing the order of time derivatives by introducing extra fields, following Ostrogradsky [18] (see also [19]). The application of such a formulation to two-dimensional gravity is the subject of this article.

The Hamiltonian formulation of higher-order theories was considered more than one and a half centuries ago by Ostrogradsky [18] for the case of non-singular systems, and was several times rediscovered by others.<sup>5</sup> Before generalization to singular systems, this approach was discussed for a few higher-order theories such as Podolsky electrodynamics [20] - [22], and scalar fields with higher-derivative couplings [23]; and was first applied to the EH action in four dimensions by Dutt and Dresden [24].

The first systematic generalization of the Ostrogradsky method to singular systems was

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<sup>3</sup> This conclusion was supported by writing down the Lagrangian for metric gravity in special coordinate systems (such as the conformal frame), where the  $\Gamma\Gamma$ -part vanishes, and then *wrongly* generalizing this to any coordinate system.

<sup>4</sup> As the action is a total derivative in the tetrad formulation, any possible transformation of tetrads is a candidate for a gauge transformation.

<sup>5</sup> The criticism that can sometimes be found on the limitation of the original Ostrogradsky results to non-singular Lagrangians is not entirely correct because he clearly stated this restriction in his analysis.

considered by Gitman, Lyakhovich and Tyutin (GLT) [25] (see also [26]), and was applied to the Hamiltonian formulation of generalized Yang-Mills theory [27], [28], and higher-derivative gravity [29].

By a suitable introduction of extra fields, this generalization of the Ostrogradsky method allows one to reformulate a problem with higher derivatives in such a way that the Dirac procedure which was originally capable of handling only theories with first-order time derivatives [3] can be used for singular, in particular, gauge theories with higher order time derivatives. Details of this generalization and its variations appearing in the literature [30], [31], and [32] are beyond the scope of this article. We instead provide an illustration of how it can be used by considering the EH action in  $d$  dimensions and then specializing to  $d = 2$  dimensions.

The Lagrangian of the (metric) EH action depends on second derivatives of the fields

$$L_g = L_g(g_{\alpha\beta}; g_{\alpha\beta,0}; g_{\alpha\beta,k}; g_{\alpha\beta,00}; g_{\alpha\beta,0k}; g_{\alpha\beta,km}) , \quad (2)$$

where we have separated time derivatives of the fields to make subsequent discussion more transparent.

If, following GLT, the additional variables

$$G_{\alpha\beta} = \dot{g}_{\alpha\beta} ; v_{\alpha\beta} = \dot{G}_{\alpha\beta} \quad (3)$$

are introduced, the Lagrangian (2) can be represented in the following way

$$L_g^v = L_g^v(g_{\alpha\beta}; G_{\alpha\beta}; g_{\alpha\beta,k}; v_{\alpha\beta}; G_{\alpha\beta,k}; g_{\alpha\beta,km}) , \quad (4)$$

where only first-order time derivative of fields appears. As it will be shown below, this Lagrangian describes the same dynamical system as that of (2) if it is supplemented by the conditions of (3)

$$S = \int \tilde{L}_g^v d^d x = \int \left[ L_g^v + \pi^{\alpha\beta} (\dot{g}_{\alpha\beta} - G_{\alpha\beta}) + \Pi^{\alpha\beta} (\dot{G}_{\alpha\beta} - v_{\alpha\beta}) \right] d^d x . \quad (5)$$

The Lagrange multipliers  $\pi^{\alpha\beta}$  and  $\Pi^{\alpha\beta}$  act as momenta conjugate to  $g_{\alpha\beta}$  and  $G_{\alpha\beta}$  respectively. Variation of this action results in the following equations of motion

$$\frac{\delta \tilde{L}_g^v}{\delta \pi^{\alpha\beta}} = \dot{g}_{\alpha\beta} - G_{\alpha\beta} = 0 , \quad (6)$$

$$\frac{\delta \tilde{L}_g^v}{\delta \Pi^{\alpha\beta}} = \dot{G}_{\alpha\beta} - v_{\alpha\beta} = 0 , \quad (7)$$

$$\frac{\delta \tilde{L}_g^v}{\delta g_{\alpha\beta}} = \frac{\partial L_g^v}{\partial g_{\alpha\beta}} - \partial_k \frac{\partial L_g^v}{\partial g_{\alpha\beta,k}} + \partial_k \partial_m \frac{\partial L_g^v}{\partial g_{\alpha\beta,km}} - \dot{\pi}^{\alpha\beta} = 0, \quad (8)$$

$$\frac{\delta \tilde{L}_g^v}{\delta G_{\alpha\beta}} = \frac{\partial L_g^v}{\partial G_{\alpha\beta}} - \partial_k \frac{\partial L_g^v}{\partial G_{\alpha\beta,k}} - \pi^{\alpha\beta} - \dot{\Pi}^{\alpha\beta} = 0, \quad (9)$$

$$\frac{\delta \tilde{L}_g^v}{\delta v_{\alpha\beta}} = \frac{\partial L_g^v}{\partial v_{\alpha\beta}} - \Pi^{\alpha\beta} = 0, \quad (10)$$

provided all variations vanish on the boundary. These are equivalent to the Lagrange equations following from variation of the original EH action (2)<sup>6</sup>

$$\begin{aligned} \frac{\delta L_g}{\delta g_{\alpha\beta}} &= \frac{\partial L_g}{\partial g_{\alpha\beta}} - \partial_0 \frac{\partial L_g}{\partial g_{\alpha\beta,0}} - \partial_k \frac{\partial L_g}{\partial g_{\alpha\beta,k}} + \\ \partial_0 \partial_0 \frac{\partial L_g}{\partial g_{\alpha\beta,00}} + \partial_0 \partial_k \frac{\partial L_g}{\partial g_{\alpha\beta,0k}} + \partial_k \partial_m \frac{\partial L_g}{\partial g_{\alpha\beta,km}} &= 0. \end{aligned} \quad (11)$$

This is easy to verify by differentiating (10) with respect to time and substituting  $\dot{\Pi}^{\alpha\beta}$  into (9). We differentiate again with respect to time to get

$$\dot{\pi}^{\alpha\beta} = \partial_0 \frac{\partial L_g^v}{\partial G_{\alpha\beta}} - \partial_0 \partial_k \frac{\partial L_g^v}{\partial G_{\alpha\beta,k}} - \partial_0 \partial_0 \frac{\partial L_g^v}{\partial v_{\alpha\beta}}. \quad (12)$$

Substituting of  $\dot{\pi}^{\alpha\beta}$  into (8) and supplementing the result with solutions of (6) and (7) finally establishes equivalence. The advantage of the new Lagrangian is that it contains only first-order time derivative of fields, permitting us to write the Hamiltonian in the usual way

$$H = \pi^{\alpha\beta} \dot{g}_{\alpha\beta} + \Pi^{\alpha\beta} \dot{G}_{\alpha\beta} - \tilde{L}_g^v, \quad (13)$$

leading to

$$H = \pi^{\alpha\beta} G_{\alpha\beta} + \Pi^{\alpha\beta} v_{\alpha\beta} - L_g^v. \quad (14)$$

This can be verified by a simple rearrangement of the terms in  $\tilde{L}_g^v$ . The fundamental Poisson brackets (PB) associated with this Hamiltonian are

$$\{g_{\alpha\beta}, \pi^{\mu\nu}\} = \{G_{\alpha\beta}, \Pi^{\mu\nu}\} = \frac{1}{2} \left( \delta_\alpha^\mu \delta_\beta^\nu + \delta_\beta^\mu \delta_\alpha^\nu \right), \quad (15)$$

and the PB for any two functionals  $A$  and  $B$  of the canonical variables is

$$\{A, B\} = \frac{\delta A}{\delta g_{\alpha\beta}} \frac{\delta B}{\delta \pi^{\alpha\beta}} + \frac{\delta A}{\delta G_{\alpha\beta}} \frac{\delta B}{\delta \Pi^{\alpha\beta}} - (A \leftrightarrow B). \quad (16)$$

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<sup>6</sup> Note that there is no coefficient “2” in the fifth term of (11) since the symmetry of double derivatives  $\partial_0 \partial_k$  has been implicitly taken into account in (2) (for an explicit form of (2) see (26) below).

As an indication of the singular nature of this Lagrangian, we observe that the equations of motion following from the Hamiltonian formulation are

$$\dot{g}_{\alpha\beta} = \{g_{\alpha\beta}, H\} = \frac{\delta H}{\delta \pi^{\alpha\beta}} = G_{\alpha\beta}, \quad (17)$$

$$\dot{G}_{\alpha\beta} = \{G_{\alpha\beta}, H\} = \frac{\delta H}{\delta \Pi^{\alpha\beta}} = v_{\alpha\beta}, \quad (18)$$

$$\dot{\pi}^{\alpha\beta} = \{\pi^{\alpha\beta}, H\} = -\frac{\delta H}{\delta g_{\alpha\beta}} = \frac{\delta L_g^v}{\delta g_{\alpha\beta}}, \quad (19)$$

$$\dot{\Pi}^{\alpha\beta} = \{\Pi^{\alpha\beta}, H\} = -\frac{\delta H}{\delta G_{\alpha\beta}} = \frac{\delta L_g^v}{\delta G_{\alpha\beta}}, \quad (20)$$

which are not equivalent to the equations of motion following from the Lagrangian formulation. This is because equation (10)

$$\frac{\partial L_g^v}{\partial v_{\alpha\beta}} - \Pi^{\alpha\beta} = 0 \quad (21)$$

is missing. If the Lagrangian were non-singular (Hessian were not zero), one would be able to solve (21) for  $v_{\alpha\beta}$  and substitute the solution back into (18) to get a consistent Hamiltonian formulation for the dynamics of the two pairs of conjugate variables  $(g_{\alpha\beta}, \pi^{\alpha\beta})$  and  $(G_{\alpha\beta}, \Pi^{\alpha\beta})$ , without any of constraints. However, for the EH Lagrangian, because it is linear in the second-order derivatives, Eq. (21) cannot be solved. Therefore Eq. (21) has to be supplemented to the Hamiltonian formulation as a set of primary constraints.

We now demonstrate this explicitly, starting in  $d$  dimensions, and then switching to two dimensions, which is the main concern of this article. If the Riemann tensor is written explicitly in terms of the metric tensor and its derivatives [24], the EH Lagrangian splits into two parts in the following way

$$L = \sqrt{(-1)^{d-1}} g R = A^{\alpha\beta\mu\nu} g_{\alpha\beta,\mu\nu} + B^{\alpha\beta\mu\nu\gamma\rho} g_{\alpha\beta,\gamma} g_{\mu\nu,\rho}, \quad (22)$$

where

$$A^{\alpha\beta\mu\nu} = \sqrt{(-1)^{d-1}} g (g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\beta} g^{\mu\nu}) \quad (23)$$

and

$$\begin{aligned} B^{\alpha\beta\mu\nu\gamma\rho} = & -\frac{1}{4} \sqrt{(-1)^{d-1}} g (g^{\alpha\beta} g^{\mu\nu} g^{\gamma\rho} - 3g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} \\ & + 2g^{\alpha\rho} g^{\beta\nu} g^{\gamma\mu} + 4g^{\alpha\gamma} g^{\mu\rho} g^{\beta\nu} - 4g^{\alpha\gamma} g^{\beta\rho} g^{\mu\nu}). \end{aligned} \quad (24)$$

After introducing the additional variables of (3), the Lagrangian becomes

$$L = \pi^{\alpha\beta} \dot{g}_{\alpha\beta} + \Pi^{\alpha\beta} \dot{G}_{\alpha\beta} - H, \quad (25)$$

with

$$\begin{aligned}
H = & \pi^{\alpha\beta} G_{\alpha\beta} + v_{\alpha\beta} \left( \Pi^{\alpha\beta} - A^{\alpha\beta 00} \right) - \left( A^{\alpha\beta 0k} + A^{\alpha\beta k0} \right) G_{\alpha\beta,k} - A^{\alpha\beta km} g_{\alpha\beta,km} \\
& - B^{\alpha\beta\mu\nu 00} G_{\alpha\beta} G_{\mu\nu} - \left( B^{\alpha\beta\mu\nu 0k} + B^{\mu\nu\alpha\beta k0} \right) G_{\alpha\beta} g_{\mu\nu,k} - B^{\alpha\beta\mu\nu km} g_{\alpha\beta,k} g_{\mu\nu,m} .
\end{aligned} \quad (26)$$

Here it can be explicitly seen how the second term in this expression leads to the aforementioned primary constraints of (10)

$$P^{\alpha\beta} \equiv \Pi^{\alpha\beta} - A^{\alpha\beta 00} , \quad (27)$$

where  $A^{\alpha\beta\mu\nu}$  is given by (23). At this point we set  $d = 2$ . By making use of the fact that in two dimensions

$$g^{11} g^{00} - g^{01} g^{01} = 1/g ,$$

one obtains the following constraints from (27)

$$P^{11} = \Pi^{11} - \frac{1}{\sqrt{-g}} \approx 0 , P^{00} = \Pi^{00} \approx 0 , P^{01} = \Pi^{01} \approx 0 . \quad (28)$$

These primary constraints  $P^{\alpha\beta}$  obviously constitute a first class system with a simple algebra of PBs

$$\{P^{\alpha\beta} , P^{\mu\nu}\} = 0 . \quad (29)$$

The secondary constraints are determined by requiring that the primary constraints be preserved in time

$$\dot{P}^{\alpha\beta} = \{P^{\alpha\beta} , H\} \equiv S^{\alpha\beta} = 0 , \quad (30)$$

leading to the following secondary constraints

$$S^{00} = -\pi^{00} - \frac{1}{2} \frac{g^{00}}{\sqrt{-g}} G_{11} + \frac{1}{2} \frac{g^{01}}{\sqrt{-g}} g_{11,1} + \frac{g^{00}}{\sqrt{-g}} g_{01,1} , \quad (31)$$

$$S^{01} = -\pi^{01} - \frac{1}{2} \frac{g^{01}}{\sqrt{-g}} G_{11} - \frac{1}{2} \frac{g^{00}}{\sqrt{-g}} g_{00,1} , \quad (32)$$

$$S^{11} = -\pi^{11} - \frac{1}{2} \frac{g^{11}}{\sqrt{-g}} G_{11} - \frac{1}{2} \frac{g^{01}}{\sqrt{-g}} g_{00,1} . \quad (33)$$

It is not difficult to show that these new constraints (31-33) commute with the set of primary constraints

$$\{P^{\alpha\beta} , S^{\mu\nu}\} = 0 , \quad (34)$$

and the PBs among secondary constraints  $S^{\alpha\beta}$  also vanish

$$\{S^{\alpha\beta}, S^{\mu\nu}\} = 0. \quad (35)$$

Thus all primary and secondary constraints are first class. Using (31-33) one can write the Hamiltonian (26) in the following way

$$H = v_{\alpha\beta} P^{\alpha\beta} - G_{\alpha\beta} S^{\alpha\beta} + \tilde{H}, \quad (36)$$

with  $\tilde{H}$  including only spatial derivatives

$$\begin{aligned} \tilde{H} = & -A^{\alpha\beta 11} g_{\alpha\beta,11} - B^{\alpha\beta\mu\nu 11} g_{\alpha\beta,1} g_{\mu\nu,1} \\ & - \left[ (A^{\alpha\beta 01} + A^{\alpha\beta 10}) G_{\alpha\beta} \right]_{,1} = - \left( \frac{g_{00,1} - 2G_{01}}{\sqrt{-g}} \right)_{,1}. \end{aligned} \quad (37)$$

The next step is to consider the time derivative of the secondary constraints to see if any new constraints arise

$$\dot{S}^{\alpha\beta} = \{S^{\alpha\beta}, H\}. \quad (38)$$

Using the form of the Hamiltonian in (36) and the fact that primary and secondary constraints are all first class, these PBs are equal to  $\{\pi^{\alpha\beta}, \tilde{H}\}$ . These latter brackets can be obtained using (13) and assuming that the fields and their spatial derivatives vanish rapidly at infinity. An alternative is to treat  $\tilde{H}$  as a spatial surface term and totally neglect it. Either method shows that these brackets vanish, leading to the closure of the Dirac procedure with six first class constraints with zero PBs among them. As there are six independent fields  $g_{\alpha\beta}$  and  $G_{\alpha\beta}$ , and there are six first class constraints, there is zero degrees of freedom. Using the Castellani procedure [33], one can restore the gauge transformation of fields by building the gauge generator  $G(\varepsilon)$  corresponding to the complete set of first class constraints. In much the same way as in [34], [35], and [17], we find that the gauge generator  $G(\varepsilon)$  is

$$G(\varepsilon) = \int dx \left( -\varepsilon_{\alpha\beta} \{P^{\alpha\beta}, H\} + \dot{\varepsilon}_{\alpha\beta} P^{\alpha\beta} \right). \quad (39)$$

This gives the following gauge transformations with arbitrary gauge parameters  $\varepsilon_{\alpha\beta}$

$$\delta g_{\alpha\beta} = \{g_{\alpha\beta}, G(\varepsilon)\} = \varepsilon_{\alpha\beta} \quad (40)$$

and

$$\delta G_{\alpha\beta} = \{G_{\alpha\beta}, G(\varepsilon)\} = \dot{\varepsilon}_{\alpha\beta}. \quad (41)$$



This is consistent with (3). The transformations of (40) are exactly the same as the transformations obtained in [8], where only the  $L_{\Gamma\Gamma}$  part of the EH Lagrangian was considered. However, there is a difference in the number of fields and the structure of the constraints in these two methods.<sup>7</sup>

We now investigate the application of the Ostrogradsky method to tetrad gravity. This merits interest since, as has already been discussed, the Lagrangian for tetrad gravity is a pure surface term in two dimensions, making its treatment using the method employed by Pirani et al [1], [2] impossible.

If one substitutes into the EH Lagrangian (22) the expression for  $g_{\mu\nu}$  in terms of tetrads

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}, \quad (42)$$

one obtains the second order Lagrangian for gravity in terms of tetrads, which can be analyzed using the Ostrogradsky method.<sup>8</sup> The result of this substitution can be written in the compact form [36]

$$L_e = \left( 2\varepsilon^{ab} \varepsilon^{\nu\rho} e_a^\mu e_{b\rho,\nu} \right)_{,\mu} = 2\varepsilon^{ab} \varepsilon^{\nu\rho} \left( e_a^\mu e_{b\rho,\nu\mu} - e_a^\sigma e_c^\mu e_{\sigma,\mu}^c e_{b\rho,\nu} \right), \quad (43)$$

where  $\varepsilon$  is the totally antisymmetric tensor  $\varepsilon^{01} = \varepsilon^{(0)(1)} = 1$ .<sup>9</sup> Introducing, as in the metric case, additional variables

$$E_\mu^a = \dot{e}_\mu^a; \quad v_\mu^a = \dot{E}_\mu^a \quad (44)$$

and performing manipulations similar to those used in obtaining (13), we get

$$\tilde{L}_e^v = \pi_a^\mu \dot{e}_\mu^a + \Pi_a^\mu \dot{E}_\mu^a - H, \quad (45)$$

with

$$H = \pi_a^\mu E_\mu^a + \Pi_a^\mu v_\mu^a - L_e^v. \quad (46)$$

We define the following fundamental PBs

$$\{e_\mu^a, \pi_b^\nu\} = \{E_\mu^a, \Pi_b^\nu\} = \delta_b^a \delta_\mu^\nu \quad (47)$$

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<sup>7</sup> In passing, we note that there are a few possible variants of the generalized Ostrogradsky method that lead to the same transformations. For example, it is enough to introduce extra fields only for those  $g_{\alpha\beta}$  components that have second-order time derivatives (only  $g_{11}$  in two dimensions). In this case, the analysis leads to four first class constraints (three primary and one secondary) for the four fields which again leads to zero degrees of freedom and the same gauge transformation as (40).

<sup>8</sup>  $\eta_{ab}$  - Minkowsky metric,  $\mu = 0, \dots, d-1$  are world indices and  $a = 0, \dots, d-1$  are tetrad indices.

<sup>9</sup> () brackets distinguish explicit values of tetrad indices.

and have

$$\begin{aligned}
L_e^v = & 2\varepsilon^{ab}e_a^0v_{b1} + 2\varepsilon^{ab}e_a^1E_{b1,1} - 2\varepsilon^{ab}e_a^0E_{b0,1} - 2\varepsilon^{ab}e_a^1e_{b0,1} \\
& - 2\varepsilon^{ab}e_a^\sigma e_c^0 E_\sigma^c E_{b1} - 2\varepsilon^{ab}e_a^\sigma e_c^1 e_{\sigma,1}^c E_{b1} + 2\varepsilon^{ab}e_a^\sigma e_c^0 E_\sigma^c e_{b0,1} + 2\varepsilon^{ab}e_a^\sigma e_c^1 e_{\sigma,1}^c e_{b0,1} .
\end{aligned} \tag{48}$$

After rearrangement of the terms in (48), one obtains

$$\begin{aligned}
H = & \Pi_a^\mu v_\mu^a - 2\varepsilon^{ab}e_a^0v_{b1} + \pi_a^\mu E_\mu^a + 2\varepsilon^{ab}e_a^\sigma e_c^0 E_\sigma^c E_{b1} - 2\varepsilon^{ab}e_a^\sigma e_c^0 e_{b0,1} E_\sigma^c \\
& + 2\varepsilon^{ab}e_a^\nu e_c^0 e_{\nu,1}^c E_{b0} - \left[ 2\varepsilon^{ab} \left( e_a^1 e_{b0,1} + e_a^1 E_{b1} - e_a^0 E_{b0} \right) \right]_{,1} .
\end{aligned} \tag{49}$$

The first two terms in this expression give rise to a set of primary constraints  $P_a^\mu$

$$P_{(0)}^0 = \Pi_{(0)}^0, P_{(1)}^0 = \Pi_{(1)}^0, P_{(0)}^1 = \Pi_{(0)}^1 + 2e_{(1)}^0, P_{(1)}^1 = \Pi_{(1)}^1 + 2e_{(0)}^0. \tag{50}$$

The PBs among these four primary constraints  $P_a^\mu$  are obviously zero. Conservation of primary constraints in time leads to the secondary constraints  $S_a^\mu$

$$\dot{P}_a^\mu = \{P_a^\mu, H\} \equiv S_a^\mu = 0. \tag{51}$$

They are given by

$$\begin{aligned}
S_{(0)}^0 = & -\pi_{(0)}^0 + 2e_{(0)}^0 e_{(0)}^0 E_1^{(1)} + 2e_{(1)}^0 e_{(0)}^0 E_1^{(0)} \\
& + 2e_{(1)}^1 e_{(0)}^0 e_{1,1}^{(0)} + 2e_{(1)}^1 e_{(1)}^0 e_{1,1}^{(1)} + 2 \left( e_{(1)}^0 e_{(1)}^0 - e_{(0)}^0 e_{(0)}^0 \right) e_{0,1}^{(1)}, \\
S_{(1)}^0 = & -\pi_{(1)}^0 + 2e_{(0)}^0 e_{(1)}^0 E_1^{(1)} + 2e_{(1)}^0 e_{(0)}^0 E_1^{(0)} \\
& + 2e_{(0)}^1 e_{(0)}^0 e_{1,1}^{(0)} + 2e_{(0)}^1 e_{(1)}^0 e_{1,1}^{(1)} + 2 \left( e_{(0)}^0 e_{(0)}^0 - e_{(1)}^0 e_{(1)}^0 \right) e_{0,1}^{(0)}, \\
S_a^1 = & -\pi_a^1 + 2e_{(0)}^1 e_a^0 E_1^{(1)} + 2e_{(1)}^1 e_a^0 E_1^{(0)} - 2e_{(0)}^1 e_a^0 e_{0,1}^{(1)} - 2e_{(1)}^1 e_a^0 e_{0,1}^{(0)}.
\end{aligned} \tag{52}$$

It is fairly easy to demonstrate that

$$\{S_a^\mu, P_b^\nu\} = 0 \tag{53}$$

for any pair of primary and secondary constraints, but calculation of PBs among secondary constraints is slightly more involved, and requires the use of test functions, leading to

$$\{S_a^\mu, S_b^\nu\} = 0. \tag{54}$$

The Hamiltonian (49) can be expressed in terms of a linear combination of constraints plus a spatial derivative term

$$H = v_\mu^a P_a^\mu - E_\mu^a S_a^\mu - \left[ 2\varepsilon^{ab} \left( e_a^1 e_{b0,1} + e_a^1 E_{b1} - e_a^0 E_{b0} \right) \right]_{,1}. \tag{55}$$

Again, there are no tertiary constraints because  $\dot{S}_a^\mu = \{S_a^\mu, H\} = 0$  due to eqs. (53) and (54). The Dirac procedure is closed, with eight first class constraints leading to zero degrees of freedom. Using the Castellani procedure [33], the generator of gauge transformation is

$$G_e(\varepsilon) = \int dx \left( -\varepsilon_\mu^a \{P_a^\mu, H\} + \dot{\varepsilon}_\mu^a P_a^\mu \right), \quad (56)$$

where  $\varepsilon_\mu^a$  are gauge parameters. This generator leads to the gauge transformation

$$\delta_e e_\mu^a = \left\{ e_\mu^a, G_e(\varepsilon) \right\} = \varepsilon_\mu^a, \quad (57)$$

for the tetrad fields.

As discussed in the introduction, the transformation (57), as well as the transformation found for the metric formulation of the EH action (40), is consistent with triviality of the equations of motion for the EH action in two dimensions. The number of gauge parameters for the metric case is three, and for the tetrad case four, the same as the number of independent fields in each case, leading to zero number of degrees of freedom according to the standard way of counting the degrees of freedom.

As has been discussed elsewhere, the tetrad representation of a spacetime leads uniquely to the metric representation [9]. However, the converse is not always true as it is not possible to uniquely determine a tetrad representation from a metric representation. For the same reason, a gauge transformation in the tetrad formulation corresponds to a unique gauge transformation in the metric formulation

$$\epsilon_{\mu\nu} = \delta_e g_{\mu\nu} = \delta_e \left( \eta_{ab} e_\mu^a e_\nu^b \right) = \eta_{ab} e_\mu^a \delta_e e_\nu^b + \eta_{ab} e_\nu^b \delta_e e_\mu^a = e_{b\mu} \varepsilon_\nu^b + e_{b\nu} \varepsilon_\mu^b, \quad (58)$$

while a gauge transformation in the metric formulation does not lead to a unique gauge transformation in the tetrad formalism even if a unique tetrad system is specified. This is because the three equations of (58) can not be solved for the four unknowns  $\varepsilon_\nu^a$  in terms of  $\epsilon_{\mu\nu}$ . In 2D the Lagrangian for tetrad gravity is a pure surface term but its Hamiltonian treatment using GLT generalization is possible. This can provide an alternative approach to demonstrate unequivalence of 3D tetrad gravity and the Chern-Simons theory [37] (which differ by a total derivative) based on their Hamiltonian formulations.

To conclude, the Hamiltonian formulation of the two-dimensional EH action as a higher-derivative theory leads to a consistent structure of constraints, and a vanishing number of degrees of freedom. The gauge transformations of (40) and (57) are different from linearized

coordinate transformations.<sup>10</sup> The number of constraints required to reproduce linearized coordinate transformations as *gauge* transformation of the two-dimensional EH action, results in there being a negative number of degrees of freedom [38] which is clearly unacceptable. Obtaining diffeomorphism invariance as a gauge transformation leads to discrepancies in the number of degrees of freedom and the number of first class constraints appearing in the Hamiltonian analysis of other two-dimensional models. As an example, a Hamiltonian analysis of a scalar field in curved spacetime  $\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$  gives five first class constraints for the four fields  $(g_{\mu\nu}, \phi)$  as reported in [39], leading to minus one number of degrees of freedom if the diffeomorphism and Weyl invariances are to be *gauge* symmetries. However, as will be reported elsewhere, the treatment of this model using the Dirac procedure removes the contradiction arising from having a negative number of degrees of freedom and leads to a gauge transformation distinct from *the linearized coordinate transformation*.

In this paper the method of GLT was employed in which only the order of temporal derivatives was decreased (the same as was done by Dutt and Dresden). An alternative way is to introduce extra fields also for spatial derivatives of fields, an approach that may be called a covariant Ostrogradsky method. Recently, the lowering of order of derivatives in covariant form was used for three dimensional topologically massive electrodynamics (TME) [40] to construct its first order formulation. This covariant Ostrogradsky method leads to consistent results, despite of the conclusion of [41], and, in particular, preserves the gauge invariance which was explicitly demonstrated using the Dirac formalism in [42].

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<sup>10</sup> The linearized coordinate transformation is  $\delta g_{\mu\nu} = -g_{\mu\lambda}\xi^\lambda_{,\nu} - g_{\nu\lambda}\xi^\lambda_{,\mu} - \xi^\lambda g_{\mu\nu,\lambda}$  [4], and the corresponding algebra of PB among constraints is known as Dirac, hypersurface-deformation, or diffeomorphism algebra. The last name actually is abuse of mathematical language because this algebra corresponds to the linearized coordinate transformation and not to the general one, which is called a diffeomorphism.

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